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On the universality of the Milan factor for $1/Q$ power corrections to jet shapes*

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Abstract: We perform the two-loop analysis of the $1/Q$ power corrections to jet-shape variables. This step is necessary for producing reliable theoretical predictions for the relative magnitudes of genuine confinement effects. We show that the rescaling factor recently derived for the thrust case (the Milan factor) remains the same for the class of observables which includes the C-parameter, invariant jet masses, jet broadening and the energy-energy correlation measure. We list the expressions which should be used for extracting $1/Q$ power effects in jet shapes. We also envisage large non-perturbative effects, characterised by *fractional* powers of Q, in certain semi-inclusive observables such as the height of the spectrum of Drell-Yan lepton pairs with invariant mass Q and transverse momentum $Q_t = 0$, and back-to-back energy flows in e^+e^- and DIS.

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Contents

1. Introduction

There is a set of jet-shape observables that contain $1/Q$ power correction terms [1, 2, 3]. These non-perturbative contributions originate from soft-gluon emission with energies of the order of the confinement scale.

The standard technology for determining these power terms relies on the introduction of a fake gluon mass. In a more systematic approach, it is the variable m^2, entering in the dispersive representation for the running QCD coupling, that plays the rôle of the power correction trigger [4, 5, 6]. Within this approach the magnitude of the power contribution is expressed in terms of a standard dimensional integral of the non-perturbative part of the QCD coupling ("non-perturbative effective coupling modification"), multiplied by an observable-dependent (easily calculable) coefficient.

In both cases the gluon decays, essential for building up the running coupling, were treated inclusively. This is known as the naive approximation. It is based on replacing the actual contribution of the final-state partons by that of their massive parent gluon. This procedure is intrinsically ambiguous since there is no unique prescription for including finite-m^2 effects into the definition of the shape variable. Moreover, Nason and Seymour [7] questioned the application of the inclusive treatment of gluon decays. They pointed out that jet shapes are not inclusive observables, and therefore that the configuration of offspring partons in the gluon decay may affect the value of the power term at next-to-leading level in α_s, which a priori is no longer a small parameter since the characteristic momentum scale is low. Both these problems can be resolved only at the two-loop level.

For the thrust case the problem was addressed recently in [8]. The two-loop analysis led to a modification of the naive prediction for the coefficient of the $1/Q$ term by a perturbatively calculable C_A- and n_f-dependent numerical factor, which has since been named "the Milan factor" [9]. A priori one might expect the Milan factors to differ for different observables.

In this paper we consider, in addition to thrust, the following infrared- and collinear-safe observables: the C-parameter, invariant jet masses and jet broadening. In near-to-two-jet kinematics (soft limit) these observables are linear in secondary particle momenta and therefore produce $1/Q$ terms.

We compute Milan factors for these observables, which rescale the naive predictions. We show that given an appropriate, consistent, definition of the naive approximation, the observables belonging to the same $1/Q$ class have a universal Milan factor, $\mathcal{M} \simeq 1.8$.

We address the problem of constructing renormalon-free answers that combine the truncated perturbative series with the power-behaving non-perturbative contributions. We choose to quantify the magnitudes of the non-perturbative contributions by means of finite-momentum integrals of the running QCD coupling α_s rather than in terms of the moments of its dispersive partner, the so-called "effective coupling" $\alpha_{\rm eff}$ [6], as suggested in [2]. The corresponding numerical "translation factor" $2/\pi$ happens to practically compensate the effect of the Milan factor. As a result, we expect the phenomenology of $1/Q$ power contributions to jet shapes to be affected only slightly.

The same Milan factor applies to the $1/Q$ power correction to the energy-energy correlation measure in e^+e^- annihilation (EEC) outside the region of back-to-back kinematics. A qualitative analysis of EEC in the special back-to-back kinematical region shows that one can expect large non-perturbative corrections characterised by fractional powers $Q^{-\gamma}$, with $\gamma < 1$. This results from an interplay between non-perturbative and perturbative effects, which is typical for semi-inclusive observables. A fractional power should also be present in the distribution of heavy lepton pairs with small transverse momenta (the Drell-Yan process).

The paper in organised as follows: in section 2 we define and describe the observables in the soft radiation limit; section 3 is devoted to the all-order resummation programme; in sections 4 and 5 we extract the leading $1/Q$ power corrections to the radiator and to the corresponding distributions; section 6 is devoted to the merging of the perturbative and non-perturbative contributions in a renormalon-free manner; we summarise the results and present conclusions in section 7.

2. Kinematics and observables

Here we consider the kinematics of multi-particle ensembles consisting of a primary quark and antiquark with 4-momenta p and $\bar{p}$, and m secondary partons k_i. Hereafter we shall consider the phase space region of soft secondary partons, soft gluons and their decay products, so that the primary p and $\bar{p}$ belong to the opposite hemispheres.

We introduce the standard light-cone Sudakov decomposition in terms of two vectors chosen in the direction of the thrust axis:

$$P^\mu + \bar{P}^\mu = Q^\mu; \quad P^2 = \bar{P}^2 = 0; \quad 2(P\bar{P}) = Q^2 \equiv 1\,.$$

The parton momenta are:

$$\begin{aligned} p &= \alpha_p \bar{P} + \beta_p P + p_t\,, \\ \bar{p} &= \alpha_{\bar{p}} \bar{P} + \beta_{\bar{p}} P + \bar{p}_t\,, \\ k_i &= \alpha_i \bar{P} + \beta_i P + k_{ti}\,. \end{aligned}$$

For a massless particle q, we have $\alpha_q \beta_q = q_t^2$. In the soft approximation, for the primary quarks we have

$$(1-\alpha_{\bar{p}}), (1-\beta_p) \ll 1\,, \qquad \beta_{\bar{p}} \simeq \bar{p}_t^2, \alpha_p \simeq p_t^2 \ll 1\,, \tag{2.1}$$

while all the secondary-parton momentum components are small. The differences $1-\alpha_{\bar{p}}$ and $1-\beta_p$ are *linear* in gluon momenta and will be taken care of; the components $\beta_{\bar{p}}$ and α_p are *quadratic* and will be neglected in what follows.

The jet shapes defined with the use of the thrust axis are T, the invariant squared mass of all the particles in one hemisphere ("jet mass" M^2) and the sum of the moduli of the momentum components transverse to the thrust axis (the jet broadening B [10]).

In terms of Sudakov variables they read:

$$1 - T = \sum_{i=1}^{N} \min(\alpha_i, \beta_i) + \alpha_p + \beta_{\bar{p}}\,, \tag{2.2}$$

$$M_R^2 = \left(\alpha_p + \sum_{i\in R} \alpha_i\right)\left(1 - \beta_{\bar{p}} - \sum_{i\in L} \beta_i\right), \tag{2.3}$$

$$2B_R = \sum_{i\in R} \left|\vec{k}_{ti}\right| + \left|\sum_{i\in R} \vec{k}_{ti}\right|. \tag{2.4}$$

Here we have defined the right hemisphere as containing the quark p, so that for a particle in the right hemisphere, $\alpha < \beta$ (correspondingly, partons with $\beta < \alpha$ contribute to M_L^2, B_L). We have also used the fact that the total transverse momentum of particles in each hemisphere is zero, which is a property of the thrust axis.

There are two types of invariant mass and broadening distributions under consideration. One can discuss the total squared mass, the total broadening, etc., which are characteristics of the whole event:

$$M_T^2 = M_R^2 + M_L^2 \,, \qquad B_T = B_R + B_L \,.$$

Alternatively, one can study the *heavy-jet* mass, the *wide-jet* broadening:

$$M_H^2 = \max\left\{M_R^2, M_L^2\right\} \,, \qquad B_W = \max\left\{B_R, B_L\right\} \,.$$

The all-order perturbative analysis has been carried out both for the total and for single-jet ("heavy", "wide") characteristics [11, 12]. As we shall see later, the non-perturbative effects in the single-jet characteristics, in general, amount to *half* of those in the *total* event observables.

The C-parameter and the energy-energy correlation function (EEC) are defined as

$$C = \frac{3}{2}\sum_{ab} p_a p_b \sin^2\theta_{ab} \,, \tag{2.5}$$

$$\mathrm{EEC}(\chi) = \sum_{ab} E_a E_b \;\delta(\cos\chi - \cos\theta_{ab}) \,, \tag{2.6}$$

where p_a and E_a are the modulus of the 3-momentum and the energy of parton a (the quark, the antiquark or one of the secondary partons); θ_{ab} is the angle between the two momenta. The sums over a, b run over all particles in an event. (Notice that within the definition of the C parameter and EEC, each pair a, b is counted twice.) In the massless particle approximation one has the sum rule

$$C = \frac{3}{2}\int_{-1}^{1} d(\cos\chi)\; \mathrm{EEC}(\chi)\;\sin^2\chi \,. \tag{2.7}$$

Now we consider each of these observables in the soft limit.

Thrust. If only one gluon is emitted, say in the right hemisphere, $\alpha_1 < \beta_1 < 1$, then

$$1 - T = \alpha_1 + \alpha_p + \beta_{\bar p} \,.$$

Since the thrust axis is such that $k_{t1}^2 = p_t^2$, we have

$$\alpha_1 = \frac{p_t^2}{\beta_1} \,, \quad \alpha_p = \frac{p_t^2}{1-\beta_1} \,, \quad \beta_{\bar p} = 0 \,.$$

The differential thrust distribution takes the form [13]

$$(1-T)\frac{d\sigma}{\sigma\,dT} = \frac{C_F\alpha_s}{\pi}\left(4\ln\frac{1}{1-T} - 3 + \mathcal{O}(1-T)\right).$$

Here the logarithmic piece comes from the soft limit neglecting the quark contribution: $\alpha_p = p_t^2 \ll \alpha_1 = p_t^2/\beta_1$, so that $1-T \simeq \alpha_1$. The constant term accounts for gluon and quark contributions in the region of relatively hard gluons: $\beta_1 \sim \beta_p \sim 1$, $\alpha_1 \sim \alpha_p$.

In higher orders the soft gluon contributions exponentiate into the double-logarithmic Sudakov form factor, while the effects of hard gluons and of quark recoil remain subleading, being down by one power of $\ln(1-T)$. However, the quark contributions to thrust, $\alpha_p + \beta_{\bar{p}}$, don't contribute at all to the $1/Q$ power-suppressed contribution, which is driven exclusively by soft gluons. Therefore hereafter we ignore these quark contributions to (2.2), though we note that the perturbative part of the answer takes them into proper account [11].

Defining

$$A = \sum_{i\in R}\alpha_i\,, \quad \bar{A} = \sum_{i\in L}\beta_i\,,$$

and neglecting the quark contributions we have

$$1 - T = A + \bar{A}\,. \tag{2.8}$$

Jet mass. It is common in the discussion of jet masses, in addition to considering the sum of the squared masses of the two hemispheres,

$$M_T^2 = M_R^2 + M_L^2\,,$$

to study the heavy jet mass,

$$M_H^2 = \max\{M_R^2, M_L^2\}\,.$$

To simplify the analysis we postpone the consideration of which hemisphere is heavier and discuss the double-differential distribution in M_R^2 and M_L^2. This will help avoid the considerable confusion that arises in a direct α_s^2 analysis of the non-perturbative effects in M_H^2. Indeed, aiming at power effects, we should treat the two hemispheres on an equal footing, since it is the *perturbative* radiation which determines which of the two jets is heavier.

The quark contributions to the jet mass (2.3) can be analysed and dropped for the study of the $1/Q$ power corrections, as in the thrust case, so that we have

$$M_R^2 = A(1-\bar{A})\,, \quad M_L^2 = \bar{A}(1-A)\,. \tag{2.9}$$

In the first order in α_s the relation between the jet masses and the thrust reads

$$1 - T \simeq M_R^2 + M_L^2 = M_T^2\,. \tag{2.10}$$

We observe that at the α_s^2 level this relation becomes non-linear. From (2.8) and (2.9) one derives

$$(1-T)\frac{1+T}{2} = M_R^2 + M_L^2 - \frac{(M_R^2 - M_L^2)^2}{2}\,. \tag{2.11}$$

This introduces corrections to (2.10) of relative order $M_{R,L}^2/Q^2$. In the region of small jet masses these effects are negligible. At the same time, averaging over, say M_L^2 in the M_R^2-distribution will produce a cross-talk correction at a relative level of $\langle M_L^2/Q^2\rangle \sim \alpha_s(Q^2)$. As a result, the power term $1/Q$ in which we are interested will also acquire a relative correction of order $\alpha_s(Q^2)$. This subleading perturbative rescaling is beyond the accuracy that we consider.

Broadening. The distribution of the jet broadening, B, is different in two respects from those of the other variables considered here. At the perturbative level, the quark recoil can not be neglected. In the first order, for example, the quark contribution to B is simply equal to that of the gluon. Hence, the factor of two in the definition of B (2.4). In higher orders an account of quark recoil becomes more complicated [12, 14]. At the non-perturbative level, broadening has an extra collinear enhancement, leading to a power correction proportional to $(\ln Q)/Q$. Broadening is the only observable currently under discussion with such a type of leading power correction. One can expect analogous (i.e. $\ln Q$-enhanced) confinement effects in the E_t distribution in hadron-hadron collisions, in the oblateness measure, etc.

C-parameter. In the soft limit, that is in the linear approximation in gluon energies, the relevant contributions to C come from counting (twice) the gluon-quark pairs,

$$C \simeq \frac{3}{4}\cdot 2\left\{\sum_i p^0 k_i^0\, \sin^2\Theta_{ip} + \sum_i \bar{p}^0 k_i^0\, \sin^2\Theta_{i\bar{p}}\right\}. \tag{2.12}$$

The gluon-gluon contributions are quadratic in gluon energies and therefore negligible. The quark-antiquark contribution can also be neglected since $\sin^2\Theta_{p\bar{p}}$ is proportional to the total gluon transverse momentum squared, and thus is also quadratic in gluon energy. Setting $p^0 = \bar{p}^0 = \frac{1}{2}$ and making use of the Sudakov variables, $k_i^0 = \frac{1}{2}(\alpha_i + \beta_i)$ and $\sin^2\Theta_{ip} = \sin^2\Theta_{i\bar{p}} = 4\alpha_i\beta_i/(\alpha_i+\beta_i)^2$, we get

$$C = 6\sum_i \frac{\alpha_i\beta_i}{\alpha_i+\beta_i}\,. \tag{2.13}$$

Energy-energy correlation. The problem of power effects in energy-energy correlations has two aspects. In the region of finite angles, $\chi \sim 1$, the leading $1/Q$ power term originates from triggering a gluon with an energy of order of the confinement momentum scale. This contribution does not involve gluon resummation and can be obtained within a fixed order analysis. It is straightforward to derive the EEC function in the soft approximation:

$$\mathrm{EEC}(\chi) = \frac{2}{\sin^3\chi} \sum_i k_{ti}\, \delta\left(1 - \frac{\alpha_i}{k_{ti}\tan\frac{\chi}{2}}\right). \tag{2.14}$$

As we shall see later, the power correction explodes at $\sin\chi \to 0$ where multi-parton resummation effects become essential.

3. Observables and the resummation programme

In what follows we concentrate on observables which require all-order soft gluon resummation to be carried out at the two-loop level. We introduce the label V as a general name for the observables $1-T$, B, C and the double jet-mass, so as to treat them simultaneously. For the broadening we consider the observable $B = B_T$. EEC away from the back-to-back region doesn't require resummation, and will be addressed at the end.

3.1. Observables

We consider the V-distributions in the region $V \ll 1$. The observables $1-T$, C and B are given as a sum of contributions $v(k_i)$ from each emitted parton k_i. They can be expressed in terms of the multi-parton emission distribution $d\sigma_n$ by the following relations

$$\begin{aligned} \frac{d\sigma}{\sigma(d\ln V)} &= \frac{d}{d\ln V}\, I(V)\,, \\ I(V) &= \sum_m \int \frac{d\sigma_m}{\sigma}\, \Theta\left(V - \sum_{i=1}^{m} v(k_i)\right), \end{aligned} \tag{3.1}$$

with m the total number of secondary partons in the final state (soft gluons or their decay products). We have $V = 1-T$, C and $2B$ (notice that B is defined in (2.4) with a factor 2, as a *half-sum* of the moduli of the transverse momenta).

For small V, which corresponds to all the final-state partons having small transverse momenta, one can write $d\sigma_m$ as a product of two factors,

$$\frac{d\sigma_m}{\sigma} = C(\alpha_{\mathrm{s}})\, dw_n\,.$$

The first is an observable-dependent coefficient factor which is a function only of $\alpha_{\mathrm{s}}(Q)$. The second is the observable-independent "evolutionary exponent", which describes the production of n small-k_t gluons off the primary quark-antiquark pair and their successive splitting into $m \geq n$ final partons. Hence

$$I(V) = C(\alpha_{\mathrm{s}}(Q))\Sigma(V,\alpha_{\mathrm{s}})\,, \qquad \Sigma(V,\alpha_{\mathrm{s}}) = \sum_n \int dw_n\, \Theta\left(V - \sum_{i=1}^{m} v(k_i)\right). \tag{3.2}$$

We note that dw_n implicitly contains a sum over m in the range $n \le m \le 2n$. The soft distribution dw_n is normalised to unity: $\sum_n \int dw_n = 1$, where the transverse momentum of each gluon is integrated up to Q. The independent emission distribution is valid only for small k_{ti}. It simplifies the treatment but mistreats the non-logarithmic region of large transverse momenta, $k_t \sim Q$, both in real and virtual terms. This is compensated by the factor $C(\alpha_s)$ in (3.2). For small V values this factor is constant. The dependence on $\ln V$ is embodied into the Σ factor.

The essential momentum scales in the coupling in Σ range from VQ to Q. The perturbative treatment requires $VQ \gg \Lambda_{\rm QCD}$. The power correction we are interested in originates from $v(k_i)Q \sim \Lambda_{\rm QCD}$.

The double-differential jet-mass distribution can be obtained via the distribution in $A, \bar{A}$ (see (2.9)) as

$$\frac{\sigma^{-1}\, d\sigma}{dM_R^2 dM_L^2} = C(\alpha_s(Q)) \int dA d\bar{A}\, \frac{d^2\Sigma(A,\bar{A})}{dA d\bar{A}}\, \delta(M_R^2 - A(1-\bar{A}))\, \delta(M_L^2 - \bar{A}(1-A))\,. \tag{3.3}$$

The double-distribution $\Sigma(A,\bar{A})$ is given by

$$\Sigma(A,\bar{A}) = \sum_n \int dw_n\, \Theta(A - \sum_{i\in R}\alpha_i)\, \Theta(\bar{A} - \sum_{i\in L}\beta_i)\,. \tag{3.4}$$

We denote it as the D-distribution (with D standing for *double*).

3.2. Soft multi-parton ensembles

To obtain the multi-parton soft-emission formula at the two-loop level it suffices to take into account a single splitting of each of the gluons radiated off the primary $q\bar{q}$ pair. It has the form

$$\begin{aligned} dw_n = {} & \frac{1}{n!}\prod_{i=1}^{n}\left\{\frac{C_F}{\pi}\frac{d\alpha_i}{\alpha_i}\frac{d^2k_{ti}}{\pi k_{ti}^2}[\alpha_s(0) + 4\pi\chi(k_{ti}^2)] + 4C_F\, d\Gamma_2(k_i,k_i')\left(\frac{\alpha_s}{4\pi}\right)^2\frac{1}{2!}M^2\right\}\times \\ & \times\exp\left\{-\frac{C_F}{\pi}\int\frac{d\alpha}{\alpha}\frac{d^2k_t}{\pi k_t^2}[\alpha_s(0)+4\pi\chi(k_t^2)] - 4C_F\int d\Gamma_2(k_1,k_2)\left(\frac{\alpha_s}{4\pi}\right)^2\frac{1}{2!}M^2\right\}. \end{aligned} \tag{3.5}$$

This expression has the following structure. The first term describes the real emission of a soft massless gluon (the term with an ill-defined $\alpha_s(0)$) and the two-loop virtual contribution to it (the renormalisation-scheme-dependent function $\chi(k_t^2)$). The last term on the first line describes the gluon splitting into a gg or $q\bar{q}$ pair. The corresponding matrix element M^2 is given in [8]. Finally, the exponential factor on the second line stands for the total virtual correction factor which ensures the normalisation of dw_n. The expression (3.5) embodies the production of up to $m = 2n$ secondary final partons.

The two-parton phase space is

$$d\Gamma_2(k_1,k_2) = \prod_{i=1}^{2} \frac{d\alpha_i}{\alpha_i}\frac{d^2k_{ti}}{\pi} = \frac{d\alpha}{\alpha}\frac{d^2k_t}{\pi}\frac{d^2q_t}{\pi}\, z(1-z)\,dz\,, \tag{3.6}$$

where $\alpha = \alpha_1 + \alpha_2$, $\vec{k}_t$ is total transverse momentum of the parent gluon, $\vec{q}_t$ is the relative "transverse angle" of the pair, and z is the fraction α_1/α. We have

$$\alpha_1 = z\alpha\,, \quad \alpha_2 = (1-z)\alpha\,, \quad \vec{k}_t = \vec{k}_{t1} + \vec{k}_{t2}\,, \quad \vec{q}_t = \frac{\vec{k}_{t1}}{z} - \frac{\vec{k}_{t2}}{1-z}\,. \tag{3.7}$$

We introduce the rapidity and the mass of the parent gluon by

$$m^2 = z(1-z)q_t^2\,; \qquad \alpha = \sqrt{k_t^2+m^2}e^{-\eta}\,, \qquad \beta = \sqrt{k_t^2+m^2}e^{\eta}\,. \tag{3.8}$$

The right (left) hemisphere corresponds to $\eta > 0$ ($\eta < 0$).

In these terms the two-parton phase space reads

$$d\Gamma_2(k_1,k_2) = dm^2\,d\eta\,\frac{d^2k_t}{\pi}\,dz\,\frac{d\phi}{2\pi}\,, \tag{3.9}$$

where ϕ is the angle between $\vec{k}_t$ and $\vec{q}_t$.

3.3. Σ–distribution

Taking advantage of the factorised structure of the multi-gluon matrix element (3.5) we introduce the source function

$$u(k) = u(\alpha,\beta) \equiv \exp(-\nu v(k))$$

to write in terms of the Mellin integral transform

$$\Theta\left(V - \sum_{i=1}^{m} v(k_i)\right) = \int \frac{d\nu}{2\pi i\,\nu}\, e^{\nu V} \prod_{i=1}^{m} u(k_i)\,.$$

Then the Σ–distribution takes the form, for $V = 1-T$, C and $2B$,

$$\Sigma(V) = \int \frac{d\nu}{2\pi i\,\nu}\, e^{\nu V}\, e^{-\mathcal{R}[u]}\,, \qquad e^{-\mathcal{R}[u]} = \sum_n \int dw_n \prod_{i=1}^{m} u(k_i)\,. \tag{3.10}$$

For the D distribution in (3.4) we have to introduce two Mellin transforms:

$$\Sigma(A,\bar{A}) = \int \frac{d\nu_1}{2\pi i\,\nu_1}\, e^{\nu_1 A} \int \frac{d\nu_2}{2\pi i\,\nu_2}\, e^{\nu_2 \bar{A}}\, e^{-\mathcal{R}[u]}\,. \tag{3.11}$$

The radiator $\mathcal{R}$ is a functional of the source $u(k)$ and has the following two-loop expression:

$$\begin{aligned}\mathcal{R}[u] = 4C_F \int_{k_t^2}^{1} \frac{d\alpha}{\alpha}\frac{d^2k_t}{\pi k_t^2}\left(\frac{\alpha_s(0)}{4\pi} + \chi(k_t^2)\right)[1-u(k)] + \\ + 4C_F \int d\Gamma_2(k_1,k_2)\left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{2!} M^2(k_1,k_2)\,[1-u(k_1)u(k_2)]\,.\end{aligned} \tag{3.12}$$

For the specific observables under consideration the source functions are:

$$\begin{aligned}
T:&\qquad u(k) = e^{-\nu\alpha}\Theta(\beta-\alpha) + e^{-\nu\beta}\Theta(\alpha-\beta)\,, \\
D:&\qquad u(k) = \Theta(\beta-\alpha)e^{-\nu_1\alpha} + \Theta(\alpha-\beta)e^{-\nu_2\beta}\,, \\
C:&\qquad u(k) = e^{-\nu c(k)}\,,\quad c(k) = 6\frac{\alpha\,\beta}{\beta+\alpha}\,, \\
B:&\qquad u(k) = e^{-\nu\sqrt{\alpha\beta}}e^{ib\sqrt{\alpha\beta}\cos\Phi}\,.
\end{aligned} \tag{3.13}$$

In the source for the distribution in total broadening, $B = B_T$, we have introduced (see [14]), in addition to the Mellin variable ν conjugate to the modulus of the transverse momentum, the two-dimensional impact parameter $\vec{b}$ conjugate to the vector transverse momentum, and we write $\vec{b}\vec{k}_t = b\sqrt{\alpha\beta}\cos\Phi$.

In (3.13) we have expressed the sources as a function of the Sudakov components α and β,

$$u(k) \;\equiv\; u(\alpha,\beta)\,,$$

where, for a massless parton, $\alpha\beta = k_t^2$.

3.4. The radiator

The various terms in $\mathcal{R}$ are ill-defined and only the sum has physical meaning. In what follows we cast $\mathcal{R}$ as sum of three contributions which, for our observables, are collinear and infrared finite. To this end we introduce a source corresponding to the parent gluon momentum $u(k_1+k_2)$ and split the source in the two-parton contribution on the second line of (3.12) as

$$1 - u(k_1)u(k_2) = [\,1 - u(k_1+k_2)\,] \;+\; [\,u(k_1+k_2) - u(k_1)u(k_2)\,]\,. \tag{3.14}$$

The expression in the first square brackets defines the "naive contribution" which treats the gluon decay inclusively. The second contribution to the radiator we refer to as the non-inclusive correction. The parent gluon momentum k_1+k_2 is "massive":

$$u(k_1+k_2) \;\equiv\; u(\alpha,\beta)\,,\qquad \alpha = \alpha_1+\alpha_2\,,\ \beta = \beta_1+\beta_2\,;\quad \alpha\beta = k_t^2+m^2\,.$$

We note that the use of the sum of 4-vectors of the offspring parton momenta in (3.14) is somewhat arbitrary. We could have used any combination of the momentum components which satisfies $v = v_1 + v_2$ in the limits of collinear and/or soft branching. The non-inclusive correction would then change correspondingly. In other words, both the naive coefficient for the power correction and the Milan factor depend on the definition of the "naive approximation," while the final answer remains unambiguous. Our prescription has the advantage of preserving those very simple numerical coefficients that are known from previous one-loop calculations.

Noticing that the variables $\vec{k}_{t1}$, $\vec{k}_{t2}$, z and m^2 are invariant with respect to Lorentz boosts in the longitudinal direction, while α is not, we conclude that the invariant matrix element M^2 does not depend on α. This allows us to perform the α-integration explicitly and represent the non-inclusive correction in the factorised form:

$$\mathcal{R}_{ni}[u] = 4C_F \int dm^2 dk_t^2 \, \frac{d\phi}{2\pi} \, dz \, \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{2!} M^2 \, \Omega_{ni} \, , \tag{3.15}$$

with the non-inclusive "trigger function"

$$\Omega_{ni} \equiv \int_{k_t^2+m^2}^{1} \frac{d\alpha}{\alpha} \left[\, u(k_1+k_2) - u(k_1)u(k_2) \,\right]. \tag{3.16}$$

The m^2-integral converges, in spite of the singular behaviour of the matrix element, $M^2 \propto 1/m^2$, because the trigger function Ω_{ni} vanishes in the collinear limit, $\Omega_{ni} \propto \sqrt{m^2}$.

The radiator then takes the form

$$\begin{aligned}\mathcal{R}[u] = \mathcal{R}_{ni} \, + \, 4C_F \int_0^1 \frac{dk_t^2}{k_t^2} \left(\frac{\alpha_s(0)}{4\pi} + \chi(k_t^2)\right) \Omega_0(k_t^2) \, + \\ + \, 4C_F \int_0^1 dm^2 \, dk_t^2 \, dz \, \frac{d\phi}{2\pi} \, \frac{M^2(k_1,k_2)}{2!} \Omega_0(k_t^2+m^2) \, ,\end{aligned} \tag{3.17}$$

where we have introduced the "naive" trigger function

$$\Omega_0(k^2) \equiv \int_{k^2}^{1} \frac{d\alpha}{\alpha} \left[\, 1 - u\left(\alpha, \, \beta = k^2/\alpha\right)\right]. \tag{3.18}$$

The inclusive integral of the two-parton matrix element on the second line of (3.17) is given, at fixed k_t, by [8]

$$\begin{aligned}\int dm^2 \, dz \, \frac{d\phi}{2\pi} \left(\frac{\alpha_s}{4\pi}\right)^2 \frac{1}{2!} M^2(k_1,k_2) = \\ = \int_0^\infty \frac{dm^2}{m^2(m^2+k_t^2)} \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{-\beta_0 + 2C_A \ln \frac{k_t^2(k_t^2+m^2)}{m^4}\right\}, \\ \beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f \, ,\end{aligned} \tag{3.19}$$

where, for $k_t^2 \ll 1$, we have replaced the actual upper limit of the m^2 integral, $m^2 < 1$, by $m^2 < \infty$ since the integral is convergent in the ultra-violet region.

The m^2-integral here is ill-defined since it diverges in the collinear two-parton limit $m^2 \to 0$. The quantity $\alpha_s(0)$ on the first line of (3.17) is also ill-defined. It combines, however, with the β_0–term in (3.19) to produce the running coupling at the scale k_t in the one-gluon emission, $\mathcal{R}_0$ (see below).

The virtual correction $\chi(k_t^2)$ also contains a collinear divergence. It can be written down in terms of the dispersive integral

$$\chi(k_t^2) = \int_0^\infty \frac{d\mu^2 \, k_t^2}{\mu^2(k_t^2+\mu^2)} \left(\frac{\alpha_s}{4\pi}\right)^2 \left\{-2C_A \ln \frac{k_t^2(k_t^2+\mu^2)}{\mu^4}\right\} + 2\left(\frac{\alpha_s}{2\pi}\right)^2 \mathcal{S} \, , \tag{3.20}$$

with $\mathcal{S}$ a scheme dependent number. In the physical scheme, in which the coupling is defined as the intensity of soft gluon radiation [15, 16], one has $\mathcal{S} = 0$. In the $\overline{\text{MS}}$ scheme, for example we have

$$\mathcal{S} = K \equiv C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5}{9} n_f\,. \tag{3.21}$$

The collinear divergences in χ at $\mu^2 \to 0$ and in the logarithmic term of (3.19) at $m^2 \to 0$, cancel for the collinear/infrared-safe observables we are dealing with. The remaining correction, the "inclusive correction" $\mathcal{R}_{in}$, is finite.

Thus the final result for the perturbative radiator $\mathcal{R}$ at the two-loop level can be cast as a sum of three terms

$$\mathcal{R} = \mathcal{R}_0 + \mathcal{R}_{in} + \mathcal{R}_{ni}\,.$$

This regularisation procedure was introduced and discussed in detail for the case of thrust in [8].

In the following we study the structure of these contributions in terms of the so-called effective coupling α_{eff} which was introduced in [6]. It is related with the standard QCD running coupling by the dispersive integral

$$\frac{\alpha_{\text{s}}(k^2)}{k^2} \;=\; \int_0^\infty dm^2 \frac{\alpha_{\text{eff}}(m^2)}{(m^2+k^2)^2}\,. \tag{3.22}$$

Its logarithmic derivative is the spectral density for α_{s}; perturbatively,

$$\frac{d}{d\ln m^2}\frac{\alpha_{\text{eff}}(m^2)}{4\pi} = -\beta_0 \left(\frac{\alpha_{\text{s}}}{4\pi}\right)^2 + \ldots\,, \qquad \alpha_{\text{eff}}(0) = \alpha_{\text{s}}(0)\,. \tag{3.23}$$

Naive contribution. The first contribution is the so-called "naive" contribution which emerges from the inclusive treatment, that is when the contributions to V of gluon decay products are replaced by that of the parent "massive" gluon. It incorporates only the term proportional to β_0 in (3.19) which, together with $\alpha_{\text{s}}(0)$ in the Born term of the radiator, builds up the running coupling in the effective one-gluon emission. We have

$$\mathcal{R}_0 \;\equiv\; 4C_F \int \frac{dm^2 dk_t^2}{k_t^2+m^2}\left\{\frac{\alpha_{\text{s}}(0)}{4\pi}\delta(m^2) - \frac{\beta_0}{m^2}\left(\frac{\alpha_{\text{s}}}{4\pi}\right)^2\right\}\Omega_0(k_t^2+m^2)\,. \tag{3.24}$$

Using (3.23) and integrating by parts we arrive at

$$\begin{aligned}\mathcal{R}_0[u] \;&=\; \frac{C_F}{\pi}\int_0^1 dm^2 \alpha_{\text{eff}}(m^2)\,\frac{-d}{dm^2}\int_0^1 \frac{dk_t^2}{k_t^2+m^2}\,\Omega_0(k_t^2+m^2) = \\ &=\; \frac{C_F}{\pi}\int_0^1 \frac{dm^2}{m^2}\alpha_{\text{eff}}(m^2)\,\Omega_0(m^2)\,.\end{aligned} \tag{3.25}$$

Inclusive correction. The second contribution is the "inclusive correction" which takes into account the logarithmic piece in (3.19) dropped in the naive treatment. If massless and massive gluons contributed equally to V, this logarithmic term would cancel completely against the second loop virtual correction χ to one-gluon emission. The mismatch is proportional to the difference of the massive and massless trigger functions. Using (3.23) and integrating by parts we obtain

$$\begin{aligned} \mathcal{R}_{in}[u] &= \frac{8C_F C_A}{\beta_0}\int \frac{dm^2}{m^2}\frac{\alpha_{\rm eff}(m^2)}{4\pi}\frac{d}{d\ln m^2}\int \frac{dk_t^2}{k_t^2+m^2}\,\ln\frac{k_t^2(k_t^2+m^2)}{m^4}\,\Omega_{in}(k_t^2,m^2)\,, \\ \Omega_{in} &\equiv \Omega_0(k_t^2+m^2)-\Omega_0(k_t^2)\,. \end{aligned} \tag{3.26}$$

Non-inclusive correction. Finally, the "non-inclusive correction" in (3.15) accounts for non-soft, non-collinear gluon decays into two partons k_1, k_2, which kinematical configurations are mistreated by the "naive" inclusive approach, as was pointed out by Nason and Seymour [7]. Using (3.23) and integrating by parts we obtain

$$\mathcal{R}_{ni}[u] = \frac{4C_F}{\beta_0}\int_0^1 \frac{dm^2}{m^2}\frac{\alpha_{\rm eff}(m^2)}{4\pi}\frac{d}{d\ln m^2}\left\{\int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^1 dz\int_0 dk_t^2\, m^2\frac{1}{2!}M^2\,\Omega_{ni}\right\}, \tag{3.27}$$

which form is suited for extracting the power correction.

In the following we discuss the three trigger functions, Ω_0, Ω_{in} and Ω_{ni}, and then the corresponding power terms, $\delta\mathcal{R}_0$, $\delta\mathcal{R}_{in}$, $\delta\mathcal{R}_{ni}$. We first consider the "Linear" observables T, D and C, and then the special case of broadening, B, (the "Log-Linear" observable).

3.4.1. The trigger functions for the Linear observables ($V = 1-T, D, C$)

Naive contribution. The "trigger functions" $\Omega_0(m^2)$ for our observables are:

$$\begin{aligned} T: &\qquad \Omega_0 = 2\int_0^{\ln m^{-1}} d\eta\left(1-\exp\left\{-m\nu e^{-\eta}\right\}\right), \\ D: &\qquad \Omega_0 = \int_0^{\ln m^{-1}} d\eta\left(1-\exp\left\{-m\nu_1 e^{-\eta}\right\}\right)+\left(1-\exp\left\{-m\nu_2 e^{-\eta}\right\}\right), \\ C: &\qquad \Omega_0 = \int_{\ln m}^{\ln m^{-1}} d\eta\left(1-\exp\left\{-m\nu\,\frac{3}{\cosh\eta}\right\}\right). \end{aligned} \tag{3.28}$$

Each Ω_0 vanishes $\propto \sqrt{m^2}$ in the small-mass limit, thus ensuring the convergence of the m^2–integration in (3.25), (3.26).

The main, perturbative, contributions to (3.25) – (3.27) come from the logarithmic integration region $1/\nu^2 < m^2 < 1$. The expression (3.25) for $\mathcal{R}_0$ reproduces then the usual double-logarithmic perturbative radiator, proportional to $\alpha_s \ln^2\nu$. The inclusive and non-inclusive contributions (3.26) and (3.27) produce two-loop subleading corrections of the order of $\alpha_s^2\ln\nu$. Actually, the standard resummation programme can only guarantee exponentiation of the terms that contain $\alpha_s^n\ln^m\nu$, with $m \geq n$.

Therefore, the perturbative corrections to $\mathcal{R}_{in}$ and $\mathcal{R}_{ni}$ do not belong to the "radiator" and are taken care of by the exact two-loop calculation (the perturbative matching procedure) [11].

The genuine non-perturbative contributions originate from the mass integration region $m^2 \ll 1/\nu^2$. In this limit, we can expand the exponents in the trigger functions (3.28) to first order in m to obtain

$$\Omega_0(m^2;V) = \delta\Omega_0 + \mathcal{O}\left(m^2\right)\,; \quad \delta\Omega_0^{(V)} = \rho^{(V)}\cdot m\,,$$

where the ρ-parameters are given by a standard rapidity integral depending on the observable under consideration:

$$\rho^{(T)} = 2\,\nu\int_0^\infty d\eta\, e^{-\eta} = 2\nu\,, \quad \rho^{(D)} = \nu_1+\nu_2\,, \quad \rho^{(C)} = \nu\int_{-\infty}^{\infty} d\eta\frac{3}{\cosh\eta} = 3\pi\nu\,. \tag{3.29}$$

Inclusive correction. In the small-m limit we have ($V=1-T,D,C$)

$$\delta\Omega_{in}^{(V)} = \rho^{(V)}\cdot\left(\sqrt{k_t^2+m^2}-k_t\right) \tag{3.30}$$

with the ρ-factors as in (3.29).

Non-inclusive correction. In the small-m limit the trigger function Ω_{ni} in (3.16) gets simplified:

$$\delta\Omega_{ni}^{(V)} = \rho^{(V)}\cdot(k_{t1}+k_{t2}-\sqrt{k_t^2+m^2})\,. \tag{3.31}$$

In the linear approximation, each source $u(k)$ produces a *transverse mass* factor ($\sqrt{k_t^2+m^2}$ for a massive gluon, and k_{ti} for a massless parton), the rest being the universal observable-dependent factor $\rho^{(V)}$ defined in (3.29). The broadening case is analogous, as discussed below.

3.4.2. The trigger functions for the Log-Linear observable ($V=2B$)

The broadening measure should be treated separately since the large-distance contribution to B is enhanced by a $\ln m^2$ factor as compared with the "Linear" T,D and C.

Naive contribution. In the case of broadening the source is independent of η for fixed (and small) transverse momentum, or rather transverse mass. As a result, an additional log-factor originates from integrating over the rapidity of a parton:

$$B: \qquad \Omega_0 = \left(1-e^{-m\nu}J_0(mb)\right)\int_{\ln m}^{\ln m^{-1}} d\eta\,. \tag{3.32}$$

In the small-m limit we obtain

$$\delta\Omega_0^{(B)} = \nu\, m \cdot \ln \frac{1}{m^2}\,.$$

This result, however, can be improved by taking into account the non-soft contribution to the gluon radiation probability. This can be done by simply using the exact quark $\to$ quark + gluon splitting function instead of the soft $d\alpha/\alpha$ spectrum in (3.5):

$$\int_{\ln m}^{\ln m^{-1}} d\eta = 2\int_{\sqrt{k_t^2+m^2}}^{1} \frac{d\alpha}{\alpha} \Longrightarrow$$

$$\Longrightarrow 2\int_{\sqrt{k_t^2+m^2}}^{1} d\alpha\, \frac{1+(1-\alpha)^2}{2\,\alpha} = \ln\frac{1}{k_t^2+m^2} - \frac{3}{2} + \mathcal{O}\left(k_t^2+m^2\right)\,. \quad (3.33)$$

With account of this correction, the small-m^2 limit of the broadening trigger function becomes

$$\delta\Omega_0^{(B)} = \nu\, m \cdot \left(\ln\frac{1}{m^2} - \frac{3}{2}\right) = \nu\, m \cdot \ln\frac{e^{-3/2}}{m^2}\,. \quad (3.34)$$

Note that the b-space resummation does not affect the leading power correction: the expansion of the exponent $\exp(i\vec{b}\vec{k}_t)$ in (3.13) lacks a term linear in k_t after integration over the azimuthal angle Φ. Thus, a small transverse momentum of a gluon, and not that of the primary quark, is relevant for the $1/Q$ power contribution to the broadening.

Inclusive correction. Given the definition of the inclusive trigger function (3.30), we substitute the naive trigger function for the broadening, (3.34), to obtain

$$\delta\Omega_{in}^{(B)} = \nu\left[\sqrt{k_t^2+m^2}\ln\frac{e^{-3/2}}{k_t^2+m^2} - k_t\ln\frac{e^{-3/2}}{k_t^2}\right]. \quad (3.35)$$

Non-inclusive correction. From the definition of the non-inclusive trigger function, (3.16), for the broadening we derive

$$\delta\Omega_{ni}^{(B)} = \nu\left(k_{t1}+k_{t2}-\sqrt{k_t^2+m^2}\right)\ln\frac{e^{-3/2}}{k_t^2+m^2}\,. \quad (3.36)$$

4. Power corrections to the radiators

To extract the leading power contributions we replace the effective coupling $\alpha_{\rm eff}(m^2)$ in the integrals for $\mathcal{R}_0$ (3.25), $\mathcal{R}_{in}$ (3.26) and $\mathcal{R}_{ni}$ (3.27) by its non-perturbative component ("effective coupling modification") $\delta\alpha_{\rm eff}(m^2)$ and employ the small-m^2 limit of the corresponding trigger functions, $\delta\Omega$.

The answers will contain the non-analytic in m^2 moments [17] of the effective coupling modification,

$$A_{2p,q} = \frac{C_F}{2\pi}\int_0^\infty \frac{dm^2}{m^2}\,(m^2)^p \ln^q\frac{m^2}{\mu^2}\,\delta\alpha_{\rm eff}(m^2)\,, \quad (4.1)$$

with μ an arbitrary momentum scale. We remind the reader that the moments analytic in m^2 vanish, that is for p integer and $q = 0$ [6, 18].

4.1. Linear observables ($V = 1-T, D, C$)

Naive contribution. We start from $\delta\mathcal{R}_0$, the power correction to the naive contribution:

$$\delta\mathcal{R}_0^{(V)} = \frac{C_F}{\pi} \int \frac{dm^2}{m^2} \delta\alpha_{\rm eff}(m^2)\, \delta\Omega_0^{(V)} = \rho^{(V)} \cdot \frac{2A_1}{Q}\,, \tag{4.2}$$

where we have restored the dimensional factor $1/Q$. Here A_1 is the $p = \frac{1}{2}, q = 0$ moment of the coupling modification (4.1):

$$A_1 \equiv A_{1,0} = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2}\, m\, \delta\alpha_{\rm eff}(m^2)\,. \tag{4.3}$$

Inclusive contribution. We have to consider the following integral in (3.26):

$$\begin{aligned} &\frac{d}{d\ln m^2} \left\{ \int_0^\infty \frac{dk_t^2}{k_t^2+m^2}\, \ln \frac{k_t^2(k_t^2+m^2)}{m^4}\, \delta\Omega_{in}^{(V)} \right\} = \\ &= \rho^{(V)} \cdot \frac{d}{d\ln m^2} \left\{ \int_0^\infty \frac{dk_t^2}{k_t^2+m^2}\, \ln \frac{k_t^2(k_t^2+m^2)}{m^4} \left(\sqrt{k_t^2+m^2} - k_t \right) \right\}. \end{aligned} \tag{4.4}$$

We have replaced here the actual upper limit of the k_t^2 integration, $Q^2 = 1$, by ∞ since the correction term is proportional to $m^2 \ln m^2$ and does not produce the leading $1/Q$ power contribution. Introducing the dimensionless integration variable $x = k_t/m$ we obtain the "inclusive" power correction in the form

$$\delta\mathcal{R}_{in}^{(T,D,C)} = \delta\mathcal{R}_0^{(T,D,C)}\, r_{in}\,, \tag{4.5}$$

with

$$r_{in} = \frac{2C_A}{\beta_0} \int_0^\infty \frac{x\,dx}{(1+x^2)(\sqrt{1+x^2}+x)}\, \ln(x^2(1+x^2)) = 3.299\, \frac{C_A}{\beta_0}\,. \tag{4.6}$$

Non-inclusive contribution. To calculate the $1/Q$ power term in the *non-inclusive* contribution in (3.27) we have to consider the multiple integral of the two-parton decay probability

$$\rho^{(V)} \frac{d}{d\ln m^2} \left\{ \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^1 dz \int_0^1 dk_t^2\, m^2 \frac{1}{2!} M^2 \left(k_{t1} + k_{t2} - \sqrt{k_t^2+m^2} \right) \right\}. \tag{4.7}$$

The linear mass-dependence can be factored out, leaving the convergent dimensionless integral which was calculated in [8]. The result reads

$$\delta\mathcal{R}_{ni}^{(T,D,C)} = \delta\mathcal{R}_0^{(C,D,C)}\, r_{ni}\,, \qquad r_{ni} = \beta_0^{-1} \left(-1.227C_A \,+\, 0.365C_A \,-\, 0.052n_f \right), \tag{4.8}$$

where the three terms originate respectively from the soft-gluon, hard-gluon and quark parts of the matrix element (see [8]).

The final results for T, D, C. Finally, the leading power contributions to the radiators of the T, D and C distributions are:

$$\delta\mathcal{R}^{(T,D,C)} = \rho^{(V)} \frac{2A_1}{Q} \mathcal{M} = \delta\mathcal{R}_0^{(T,D,C)} \mathcal{M}, \tag{4.9}$$

with the ρ factors given in (3.29). This answer differs from the naive predictions $\mathcal{R}_0^{(V)}$ defined in (4.2) by the *universal* "Milan factor",

$$\begin{aligned} \mathcal{M} \; &= \; 1 + r_{in} + r_{ni} = 1 + \beta_0^{-1} (2.437 C_A \; - \; 0.052 n_f) = \\ &= 1.795\,(1.665) \qquad\qquad \text{for } n_f = 3\,(0)\,. \end{aligned} \tag{4.10}$$

We combine the Milan factor with the A_1 moment to define a small parameter

$$\mathcal{P} \;\equiv\; \frac{2A_1}{Q} \mathcal{M}\,. \tag{4.11}$$

The non-perturbative contribution to the radiator becomes $\delta\mathcal{R}^{(V)} = \rho^{(V)} \mathcal{P}$.

4.2. The case of broadening

The answer for broadening contains two moments: the $\ln Q$–enhanced term proportional to A_1 and a new log-moment (4.1) with $p = \frac{1}{2}, q = 1$:

$$A_{1,1} \equiv A_1' = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2}\, m \ln \frac{m^2}{\mu^2}\, \delta\alpha_{\rm eff}(m^2)\,. \tag{4.12}$$

Naive contribution. The naive power contribution to the B–radiator reads, see (3.25), (3.34),

$$\delta\mathcal{R}_0^{(B)} = \nu \cdot \left[\frac{2A_1}{Q} \left(\ln \frac{Q^2}{\mu^2} - \frac{3}{2} \right) - \frac{2A_1'}{Q} \right]. \tag{4.13}$$

Since the value A_1' is not apriori known, we can eliminate it by choosing μ which then becomes a new non-perturbative parameter.

Inclusive contribution. For the B-distribution (4.4) becomes

$$\nu \frac{d}{d\ln m^2} \left\{ \int_0^\infty \frac{dk_t^2}{k_t^2 + m^2} \ln \frac{k_t^2 (k_t^2 + m^2)}{m^4} \left(\sqrt{k_t^2 + m^2} \ln \frac{Q^2 e^{-3/2}}{k_t^2 + m^2} - k_t \ln \frac{Q^2 e^{-3/2}}{k_t^2} \right) \right\}. \tag{4.14}$$

The answer reads, see (3.35)

$$\delta\mathcal{R}_{in}^{(B)} = \nu \left\{ r_{in} \frac{2A_1 (\ln Q^2/\mu^2 - 3/2 - 2) - 2A_1'}{Q} + r_{in}' \frac{2A_1}{Q} \right\}, \tag{4.15}$$

where

$$r_{in}' = \frac{2C_A}{\beta_0} \int_0^\infty \frac{x\,dx}{(1+x^2)} \left[x \ln x^2 - \sqrt{1+x^2} \ln(1+x^2) \right] \ln(x^2(1+x^2)) = -22.751 \frac{C_A}{\beta_0}\,. \tag{4.16}$$

Non-inclusive contribution. The non-inclusive power correction to the broadening radiator is obtained by using (3.36), the small-m part of the trigger function, in (3.15):

$$\delta\mathcal{R}_{ni}^{(B)} = \nu \left\{ r_{ni} \frac{2A_1(\ln Q^2/\mu^2 - 3/2 - 2) - 2A_1'}{Q} + r_{ni}' \frac{2A_1}{Q} \right\}, \tag{4.17}$$

where r_{ni}' is given by the dimensionless integral similar to that for r_{ni} in (4.8) but with an additional logarithmic factor:

$$r_{ni}' = (4\beta_0)^{-1} \int \frac{d\phi}{2\pi} \int dz \int \frac{d^2k_t}{\pi} (m^2 M^2) \frac{k_{t1} + k_{t2} - \sqrt{k_t^2 + m^2}}{m} \ln \frac{m^2}{k_t^2 + m^2}. \tag{4.18}$$

Its numerical value is

$$r_{ni}' = \beta_0^{-1} \left(4.808 C_A \; - \; 0.884 C_A \; + \; 0.116 n_f \right), \tag{4.19}$$

where the three terms, as before, correspond to the soft-gluon, hard-gluon and quark contributions.

The final result for B. The full power term in the broadening radiator is obtained by combining the three contributions. One reconstructs the Milan factor and finds

$$\delta\mathcal{R}^{(B)} = 2\nu \cdot \frac{2\mathcal{M}}{Q} \left[A_1 \left(\ln \frac{Q}{\mu} - \xi \right) - \left(\tfrac{1}{2} A_1' + A_1 \right) \right], \tag{4.20}$$

where

$$\xi = \frac{3}{4} - \frac{2 + r_{in}' + r_{ni}'}{2\mathcal{M}} = 1.692 + \frac{0.0765\, n_f}{C_A/3 - 0.0392 n_f} = 1.930\,(1.692) \qquad \text{for}\, n_f = 3\,(0)\,. \tag{4.21}$$

Here we have introduced the combination of the moments, $\frac{1}{2}A_1' + A_1$, which will be of convenience later on, see section 6.

Defining also the parameter $\mathcal{P}'$, we can write

$$\delta\mathcal{R}^{(B)} = 2\nu \cdot \left[\mathcal{P} \left(\ln \frac{Q}{\mu} - \xi \right) - \mathcal{P}' \right], \qquad \mathcal{P}' \equiv \frac{A_1' + 2A_1}{Q} \mathcal{M}\,. \tag{4.22}$$

The new unknown parameter, $\mathcal{P}'$ can be traded for the scale of the logarithm:

$$\delta\mathcal{R}^{(B)} = 2\nu \cdot \mathcal{P} \, \ln \frac{Q}{Q_B}\,. \tag{4.23}$$

The scale Q_B is observable-dependent and reads

$$\ln \frac{Q_B}{\mu} \; = \; \frac{\frac{1}{2}A_1' + A_1}{A_1} + \xi = \frac{\mathcal{P}'}{\mathcal{P}} + \xi\,. \tag{4.24}$$

5. Power effects in the distributions

The leading power contributions to the radiators are proportional to the corresponding Mellin variable ν, see (4.9), (4.23). Inserting this into (3.10) one obtains a *shift* of the corresponding perturbative distribution [19] proportional to the $\mathcal{P}$ parameter (4.11).

We arrive, for small $V = 1-T, D, C, B$ at

$$\frac{d\sigma}{dT}(1-T) = \left[\frac{d\sigma}{dT}(1-T-2\mathcal{P})\right]_{\mathrm{PT}} \tag{5.1}$$

$$\frac{d\sigma}{dA\,d\bar{A}}(A,\bar{A}) = \left[\frac{d\sigma}{dA\,d\bar{A}}\left(A-\mathcal{P},\,\bar{A}-\mathcal{P}\right)\right]_{\mathrm{PT}} \tag{5.2}$$

$$\frac{d\sigma}{dC}(C) = \left[\frac{d\sigma}{dC}(C-3\pi\mathcal{P})\right]_{\mathrm{PT}} \tag{5.3}$$

$$\frac{d\sigma}{dB}(B) = \left[\frac{d\sigma}{dB}(B-\mathcal{P}\ln(Q/Q_B))\right]_{\mathrm{PT}}. \tag{5.4}$$

These expressions are valid as long as V, though numerically small, stays at the same time much larger than Λ_{QCD}/Q, that is $V \gg \mathcal{P}$. The kinematical region $V < \mathcal{P}$ is dominated by confinement physics: the jets get squeezed down to small-mass (exclusive) hadron systems.

By expanding the general answer for the distributions in the Taylor series in $\ln V$,

$$f(V) = f_{\mathrm{PT}}(V - c_V\mathcal{P}) = f_{\mathrm{PT}}(V) - \frac{c_V\mathcal{P}}{V}f'_{\mathrm{PT}}(V) + \tfrac{1}{2}\left(\frac{c_V\mathcal{P}}{V}\right)^2(f''_{\mathrm{PT}} - f'_{\mathrm{PT}}) + \dots\,, \tag{5.5}$$

with $g'(V) \equiv dg/d\ln V$, we observe that the true expansion parameter is $\mathcal{P}/V$, rather than $\mathcal{P}$. Such an enhancement at small V is a common feature of the power contributions. For example, the $1/Q^2$ power terms in DIS structure functions and e^+e^- fragmentation functions are relatively enhanced near the phase space boundary as $(1-x)^{-1}$, where x is a usual Bjorken (DIS) or Feynman (e^+e^-) variable [6].

Strictly speaking, within our approach only the leading power correction was kept under control. Therefore we should have dropped the rest of the series, starting from the quadratic term $\mathcal{O}\left((\mathcal{P}/V)^2\right)$. In spite of this we prefer to present the result in terms of the shifted perturbative spectra, because this form is easier to implement practically. The subleading power contributions become comparable with the leading one for $V \gtrsim \mathcal{P}$ and remain to be studied.

5.1. Jet mass distribution

The invariant jet mass distribution(s) can be derived from the double differential distribution (5.2) with use of the relation (3.4). Generally speaking, the relation between the Sudakov variables A, $\bar{A}$ and the jet masses, M_L^2, M_R^2, is non-linear:

$$dA\,d\bar{A} = \frac{dM_R^2\,dM_L^2}{J}\,, \tag{5.6}$$

$$J \equiv 1 - A - \bar{A} = \sqrt{1 - 2(M_R^2 + M_L^2) + (M_R^2 - M_L^2)^2} = 1 + \mathcal{O}\left(M^2\right)\,. \tag{5.7}$$

The relations inverse to (2.9) are

$$A = M_R^2\left(1 + M_L^2\cdot\epsilon\right) \qquad \bar{A} = M_L^2\left(1 + M_R^2\cdot\epsilon\right) \tag{5.8}$$

$$\epsilon \equiv \frac{2}{1 - M_R^2 - M_L^2 + J} = 1 + \mathcal{O}\left(M^2\right). \tag{5.9}$$

Calculating the mass distribution in terms of the shifted A, $\bar{A}$ perturbative spectrum in (3.4) amounts to shifting the M^2 values according to

$$\frac{d^2\sigma}{dM_R^2\,dM_L^2}(M_R^2, M_L^2) = \left[\frac{d^2\sigma}{dM_R^2\,dM_L^2}(M_R^2 - \mathcal{P}J + \mathcal{P}^2,\ M_L^2 - \mathcal{P}J + \mathcal{P}^2)\right]_{\mathrm{PT}}\,.$$

y The quadratic terms should be dropped within our accuracy. As for the linear power shift, in the small-mass limit, $M_R^2, M_L^2 \ll 1$, we can approximate $\mathcal{P}J \simeq \mathcal{P}$. Therefore, in the small M^2 region we can simplify the answer as

$$\frac{d^2\sigma}{dM_R^2\,dM_L^2}(M_R^2, M_L^2) = \left[\frac{d^2\sigma}{dM_R^2\,dM_L^2}(M_R^2 - \mathcal{P},\ M_L^2 - \mathcal{P})\right]_{\mathrm{PT}}\,. \tag{5.10}$$

Notice, however, that in the calculation of the *mean* squared invariant mass(es) the Jacobian factor will induce a perturbative correction to the non-perturbative shift of the order of $\mathcal{P}\cdot\alpha_{\mathrm{s}}(Q^2)$, since $\langle J\rangle \sim 1 + \langle M^2\rangle = 1 + \mathcal{O}\left(\alpha_{\mathrm{s}}\right)$.

5.2. Power effects in whole-event and single-jet shapes

Some shape observables characterise a single jet rather than the whole (2-jet) ensemble. Among these are such characteristics as the heavy-jet mass, the wide-jet broadening, etc., to be compared with the mean jet mass, and the total B. Within the first order analysis aiming at power effects, that very gluon that induced the power-behaving shift in the jet distribution, was making the jet under consideration "heavier" ("wider") than its partner jet devoid of any radiation. This resulted in a strange unphysical picture in which all the non-perturbative effects were contained in the heavy/wide jet alone [1]. A proper treatment of the heavy-light relationship has been given by Akhoury and Zakharov in [20].

To illustrate the solution of the puzzle, let us take the total and heavy-jet mass distributions as an example. For the distribution in $M_T^2 = M_R^2 + M_L^2$ we immediately obtain the *doubled shift* of the corresponding perturbative expression, as compared with (5.10):

$$\frac{d\sigma}{dM_T^2}(M_T^2) \equiv \int_0^\infty dM_R^2 \int_0^\infty dM_L^2 \delta(M_T^2 - M_R^2 - M_L^2) \left[\frac{d^2\sigma}{dM_R^2 dM_L^2}(M_R^2 - \mathcal{P}, M_L^2 - \mathcal{P})\right]_{\rm PT} =$$
$$= \int\int dM_1^2\, dM_2^2\, \delta(M_T^2 - 2\mathcal{P} - M_1^2 - M_2^2) \left[\frac{d^2\sigma}{dM_R^2\, dM_L^2}(M_1^2, M_2^2)\right]_{\rm PT} \equiv$$
$$\equiv \left[\frac{d\sigma}{dM_T^2}(M_T^2 - 2\mathcal{P})\right]_{\rm PT}. \tag{5.11}$$

To calculate the *heavy-jet* mass distribution one needs to perform the integration over the smaller of the two jet masses (for definiteness, we take $M_H = M_R > M_L$ and double the answer):

$$\frac{d\sigma}{dM_H^2}(M_H^2) \equiv 2\int^{M_H^2} dM_L^2 \left[\frac{d^2\sigma}{dM_R^2\, dM_L^2}(M_H^2 - \mathcal{P},\, M_L^2 - \mathcal{P})\right]_{\rm PT} =$$
$$= 2\int^{M_H^2 - \mathcal{P}} dM_1^2 \left[\frac{d^2\sigma}{dM_R^2\, dM_L^2}(M_H^2 - \mathcal{P},\, M_1^2)\right]_{\rm PT} \equiv$$
$$\equiv \left[\frac{d\sigma}{dM_H^2}(M_H^2 - \mathcal{P})\right]_{\rm PT}. \tag{5.12}$$

Recall that the perturbative answer for the integrated heavy-jet mass spectrum is essentially the *squared* single-jet distribution [11],

$$\Sigma_H(M_H^2) = \left(\Sigma_{\text{one-jet}}(M_H^2)\right)^2,$$

where $\Sigma_{\text{one-jet}}$ describes a single-inclusive distribution in, say, M_R^2, integrated over M_L^2 without constraint. This corresponds to putting $\nu_2 = 0$ in (3.11) and (3.13). Thus we conclude that while the total-mass distribution acquires the doubled shift, $2 \cdot \mathcal{P}$, ($\nu_1 = \nu_2$ in the Mellin transformation language), the heavy-mass spectrum is sensitive to confinement effects in a single jet only ($\nu_2 = 0$), and therefore the corresponding perturbative expression gets shifted by $1 \cdot \mathcal{P}$.

The same pattern holds for the total- versus wide-jet broadening:

$$\frac{d\sigma}{dB_W}(B_W) = \left[\frac{d\sigma}{dB_W}\left(B_W - \tfrac{1}{2}\cdot \mathcal{P}\ln(Q/Q_B)\right)\right]_{\rm PT}, \tag{5.13}$$

to be compared with the twice larger shift in the total broadening spectrum (5.4).

5.3. Energy-energy correlation

At the perturbative level, the gluon-quark and quark-quark contributions to the energy-energy correlation function EEC(χ) defined in (2.6) are comparable. As far as the power effects are concerned, the leading $1/Q$ contribution to EEC originates from the correlation between one of the primary quarks and a very soft gluon, $\mathrm{EEC}_{qg}^{\rm NP}$. The gluon has an energy of order of the confinement scale, $\omega \sim k_t \sim \Lambda_{\rm QCD}$. It is easy to see that this power correction to EEC acquires the same universal Milan factor as the T, D, C and B distributions:

$$\mathrm{EEC}(\chi) = \mathrm{EEC}^{\mathrm{PT}}(\chi) + \frac{4}{\sin^3\chi}\,\frac{A_1\,\mathcal{M}}{Q} = \mathrm{EEC}^{\mathrm{PT}}(\chi) + \frac{2\mathcal{P}}{\sin^3\chi}\,. \tag{5.14}$$

While we will not present here explicit results for the quark-quark contribution, $\mathrm{EEC}^{\mathrm{NP}}_{qq}$, it is straightforward to show that it is of the form

$$\mathrm{EEC}^{\mathrm{NP}}_{qq} \sim \frac{A_{2,1}}{Q^2\sin^4\chi}\,.$$

Both the quark-gluon and quark-quark contributions are more singular at small $\sin\chi$ than the perturbative part of EEC:

$$\frac{\mathrm{EEC}^{\mathrm{NP}}_{qg}(\chi)}{\mathrm{EEC}^{\mathrm{PT}}(\chi)} \sim \frac{A_1}{Q\,\sin\chi}\,, \qquad \frac{\mathrm{EEC}^{\mathrm{NP}}_{qq}(\chi)}{\mathrm{EEC}^{\mathrm{PT}}(\chi)} \sim \frac{A_{2,1}}{Q^2\,\sin^2\chi}\,. \tag{5.15}$$

This, as we have discussed above, is a standard feature of power-suppressed contributions. The power-suppressed contribution from the gluon-gluon correlation may contribute as much as the quark-quark component, but it requires more study.

In the region of very small $\sin\chi$ values, all-order effects in the perturbative quark-quark correlation become essential, which makes the perturbative correlation finite in the back-to-back configuration, that is for $\chi = \pi$ [21, 22]. Qualitatively, what happens is that the quark and antiquark are not exactly back-to-back, with the typical acollinearity angle, $\bar{\chi}$, being given by

$$\sin\bar{\chi} = (Q\bar{b})^{-1}\,. \tag{5.16}$$

Here $\bar{b} = \bar{b}(Q)$ is the characteristic impact parameter which determines the perturbative double-logarithmic distribution. A naive estimate gives:

$$\bar{b}(Q) \sim \frac{1}{Q}\cdot\left(\frac{Q}{\Lambda_{\mathrm{QCD}}}\right)^{\beta_0/(\beta_0+4C_F)} \propto Q^{-0.372} \qquad \text{for } n_f = 3\,. \tag{5.17}$$

The acollinearity of the quarks leads to the freezing of the singularities in (5.15) when $\pi - \chi \lesssim \bar{\chi}$. For the back-to-back correlation we therefore expect

$$\frac{\mathrm{EEC}^{\mathrm{NP}}_{qg}(\pi)}{\mathrm{EEC}^{\mathrm{PT}}(\pi)} \sim A_1\,\bar{b}(Q)\,, \qquad \frac{\mathrm{EEC}^{\mathrm{NP}}_{qq}(\pi)}{\mathrm{EEC}^{\mathrm{PT}}(\pi)} \sim A_{2,1}\,\bar{b}^2(Q)\,. \tag{5.18}$$

We expect the same fractional powers of Q for the non-perturbative corrections to the value of the energy-weighted particle flow in the current fragmentation region in DIS, in the photon direction ($q_t = 0$). In addition, for the value of $d\sigma/dM^2\,dq_t^2$, the height of the plateau in the transverse momentum distribution of Drell-Yan pairs (W, Z bosons) at $q_t = 0$, we expect a contribution which is quadratic in $\bar{b}(Q)$,

$$\left.\frac{d\sigma}{dQ^2 dq_t^2}\right|_{q_t=0} = \left[\left.\frac{d\sigma}{dQ^2 dq_t^2}\right|_{q_t=0}\right]_{\mathrm{PT}} \exp\left\{\tfrac{1}{2}A_{2,1}\cdot\bar{b}^2(Q)\right\}.$$

A more detailed analysis reveals that the estimates for the values of the exponents are too naive, but that non-integer power effects are present [23].

6. Merging the PT and NP contributions

We now discuss how to obtain the full expressions for the distributions by merging the perturbative and the non-perturbative contributions. We follow the procedure suggested in [2]. The answer we shall arrive at will differ, however, from that of [2] by a factor normalising the power terms. Part of it is the Milan factor we have discussed above. As we shall shortly see, an additional factor, $2/\pi$, arises due to the "translation" from the effective coupling $\alpha_{\rm eff}$ (which we use to trigger the large-distance power contributions within the dispersive approach of [6]) to the standard QCD coupling $\alpha_{\rm s}$.

For the radiators of the $V = 1-T, D, C$ distributions which contain $1/Q$ power terms we have obtained

$$\mathcal{R}(Q^2) = \mathcal{R}^{\rm PT}(Q^2) + \rho^{(V)}\mathcal{P}\,, \tag{6.1}$$

where $\mathcal{P}$ is given in (4.11) in terms of the moment A_1 of $\delta\alpha_{\rm eff}$.

This representation is symbolic. Indeed, its perturbative part is given by an expansion which is factorially divergent. Meanwhile, the non-perturbative "effective coupling modification", $\delta\alpha_{\rm eff}$, also implicitly contains an ill-defined all-order subtraction of the pure perturbative part off the full effective coupling $\alpha_{\rm eff}$.

These two problems are of a similar nature. The uncertainty can be resolved at the price of introducing an "infrared matching scale", μ_I. To this end we represent the answer as a sum of three terms:

$$\mathcal{R}(Q) = \mathcal{R}^{\rm PT}(Q;N) + \mathcal{R}^{\rm NP}(Q,\mu_I) - \mathcal{R}^{\rm merge}(Q,\mu_I;N) \;+\; \mathcal{O}\left(\alpha_{\rm s}^{N+1}\right)\,, \tag{6.2}$$

with N the order at which the perturbative expansion is truncated.

The factorially growing PT-expansion terms (infrared renormalons) in the first (PT) and the last ("merging") pieces in (6.2) cancel. The μ_I-dependence in the second and the third contributions approximately cancels as well, the cancellation improving for larger N.

The structure of (6.2) is general; the same approach can be applied to incorporate the leading power contributions into different observables p, q. We proceed with the radiators for $V = 1-T, D, C$ distributions ($p = \frac{1}{2}, q = 0$) taken as an example of the general matching procedure (6.2). To this end we split the running coupling, formally, into two pieces,

$$\alpha_{\rm s}(k^2) = \alpha_{\rm s}^{\rm PT}(k^2) \;+\; \alpha_{\rm s}^{\rm NP}(k^2)\,.$$

Here $\alpha_{\rm s}^{\rm PT}(k^2)$ should be understood not as a mere function but rather as an operational procedure: its momentum integrals should be evaluated *perturbatively*, that is order by order in perturbation theory as a series in $\alpha_{\rm s} = \alpha_{\rm s}(Q^2)$. Its non-perturbative counterpart, $\alpha_{\rm s}^{\rm NP}(k^2)$, corresponds to the $\delta\alpha_{\rm eff}(m^2)$ part of the effective coupling. It is supposed to be a rapidly falling function in the *ultra-violet* region [6, 24]. In the *infrared* momentum region neither of the two is a well-defined object. However, the full physical coupling $\alpha_{\rm s}(k^2)$ is supposed to be finite down to $k^2 \to 0$. Its infrared momentum integrals will play the rôle of phenomenological parameters. To define these parameters we first use the formal dispersive relation (3.22) for $\alpha_{\rm s}^{\rm NP}$ and its partner, $\delta\alpha_{\rm eff}$, to deduce

$$\int_0^\infty dk\, \alpha_{\rm s}^{\rm NP}(k^2) = \frac{\pi}{4}\int_0^\infty \frac{dm^2}{m^2}\, m\, \delta\alpha_{\rm eff}(m^2)\,. \tag{6.3}$$

Then, for A_1 we have

$$A_1 = \frac{2C_F}{\pi^2}\int_0^\infty dk\, \alpha_{\rm s}^{\rm NP}(k^2) = \frac{2C_F}{\pi^2}\int_0^{\mu_I} dk\, \alpha_{\rm s}^{\rm NP}(k^2) + \mathcal{O}\left(\frac{\mu_I}{Q}\alpha_{\rm s}^{\rm NP}(\mu_I^2)\right)\,. \tag{6.4}$$

Here we have introduced a scale μ_I above which the coupling is well matched by its logarithmic perturbative expression, so that the error proportional to $\mu_I \alpha_{\rm s}^{\rm NP}(\mu_I^2)$ induced by truncation can be neglected. Now we substitute

$$\alpha_{\rm s}^{\rm NP}(k^2) = \alpha_{\rm s}(k^2) - \alpha_{\rm s}^{\rm PT}(k^2)$$

into (6.4) and approximate A_1 as

$$A_1 \simeq \frac{2C_F}{\pi^2}\int_0^{\mu_I} dk\, \alpha_{\rm s}(k^2) - \frac{2C_F}{\pi^2}\int_0^{\mu_I} dk\, \alpha_{\rm s}^{\rm PT}(k^2)\,. \tag{6.5}$$

The first contribution can be expressed in terms of a (μ_I-dependent) phenomenological parameter α_0,

$$\int_0^{\mu_I} dk\, \alpha_{\rm s}(k^2) \;\equiv\; \mu_I \cdot \alpha_0(\mu_I)\,. \tag{6.6}$$

Thus, the power contribution in (6.1) splits into two terms:

$$\delta\mathcal{R}^{(V)} = \rho^{(V)}\,\mathcal{P} = \mathcal{R}^{\rm NP}(Q,\mu_I) - \mathcal{R}^{\rm merge}(Q,\mu_I)\,. \tag{6.7}$$

The first term is the integral over the infrared region of the full coupling,

$$\mathcal{R}^{\rm NP} = \rho^{(V)}\,\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I\,\alpha_0}{Q}\,. \tag{6.8}$$

The subtraction term is given by the integral of the perturbative coupling,

$$\mathcal{R}^{\rm merge}(Q,\mu_I) = \rho^{(V)}\frac{4C_F}{\pi^2}\,\frac{\mathcal{M}}{Q}\int_0^{\mu_I} dk\, \alpha_{\rm s}^{\rm PT}(k^2)\,. \tag{6.9}$$

This perturbative subtraction contribution is obtained by substituting for $\alpha_{\rm s}^{\rm PT}(k^2)$ its perturbative series,

$$\alpha_{\rm s}^{\rm PT}(k^2) = \sum_{\ell=1}^{\infty} \alpha_{\rm s}^{\ell}\, P_\ell\left(\ln\frac{\mu_R}{k}\right)\,, \qquad \alpha_{\rm s} \equiv \alpha_{\rm s}(\mu_R^2)\,. \tag{6.10}$$

Here μ_R is the renormalisation scale which hereafter we set equal to Q. The representation (6.7) is still symbolic, since the all-order perturbative subtraction diverges. Indeed, integrating (6.10) term by term one finds

$$\int_0^{\mu_I} dk\, \alpha_s^{\rm PT}(k^2) = \mu_I \cdot \alpha_s \sum_{\ell=0}^{\infty} \left(\beta_0 \frac{\alpha_s}{2\pi}\right)^\ell C_\ell\,, \tag{6.11}$$

with C_ℓ being factorially growing coefficients. This behaviour is general. Taking, for the sake of illustration, the one-loop coupling, in which case (6.10) is a simple geometric series,

$$\alpha_s^{\rm PT}(k^2) = \alpha_s \sum_{\ell=0}^{\infty} \left(\beta_0 \frac{\alpha_s}{2\pi} \ln\frac{Q}{k}\right)^\ell , \quad \beta_0 = \frac{11N_c}{3} - \frac{2n_f}{3}\,, \tag{6.12}$$

one obtains

$$C_\ell = \frac{1}{Q}\int_0^{\mu_I} dk \left(\ln\frac{Q}{k}\right)^\ell = \int_{\ln(Q/\mu_I)}^{\infty} dt\, e^{-t}\, t^\ell\,, \qquad t = \ln\frac{Q}{k}\,. \tag{6.13}$$

For finite ℓ we have $C_\ell \simeq (\ln(Q/\mu_I))^\ell$, while for $\ell \gg \ln(Q/\mu_I)$ the coefficients start to grow as $C_\ell \sim \ell!$ (infrared renormalon).

The perturbative part of the answer, $\mathcal{R}^{\rm PT}$ in (6.1), bears the same (infrared renormalon) divergence. As we have anticipated above, the factorial behaviours of the coefficients of these two perturbative expansions cancel. Indeed the corresponding coefficients in $\mathcal{R}^{\rm PT}$ for large ℓ are given by integrals identical to (6.13) but with the upper limit taken as $k \le Q$ $(t \ge 0)$. Therefore the factorial divergence in $\mathcal{R}^{\rm PT}$ originating from the low-momentum integration region in the Feynman diagrams, $k \sim Qe^{-\ell}$, is subtracted off exactly by $\mathcal{R}^{\rm merge}$. The mismatch vanishes at large $\ell \gg \ln\frac{Q}{\mu_I}$. Due to this cancellation, the difference $(\mathcal{R}^{\rm PT} - \mathcal{R}^{\rm merge})$ is well defined perturbatively, so we can truncate the perturbative expansions for both $\mathcal{R}^{\rm PT}$ and $\alpha_s^{\rm PT}(k^2)$ in $\mathcal{R}^{\rm merge}$ at some finite N^{th} order.

The merging piece truncated at $\ell = N-1$ takes the form

$$\mathcal{R}^{\rm merge}(Q,\mu_I;N) = \rho^{(V)}\,\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\,\alpha_s \sum_{\ell=0}^{N-1}\left(\beta_0\frac{\alpha_s}{2\pi}\right)^\ell C_\ell\,. \tag{6.14}$$

The final answer reads

$$\mathcal{R}(Q) = \mathcal{R}^{\rm PT}(Q;N) + \rho^{(V)}\,\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\left(\alpha_0 - \alpha_s\sum_{\ell=0}^{N-1}\left(\beta_0\frac{\alpha_s}{2\pi}\right)^\ell C_\ell\right) + \Delta_{\mathcal{R}}(Q,\mu_I;N)\,. \tag{6.15}$$

The answer contains two components, a perturbative piece with a smooth logarithmic Q-dependence, $\mathcal{R}^{\rm PT} = \mathcal{R}^{\rm PT}(\alpha_s(Q^2))$, and a steep power term $\mathcal{R}_{\rm NP} - \mathcal{R}^{\rm merge} \propto \mu_I/Q$.

Correspondingly, the error $\Delta_{\mathcal{R}}$ of the representation (6.15) contains two separate components.

For a moderate N the error in the perturbative part is obviously given by

$$\Delta_{\mathcal{R}}^{\mathrm{PT}} = \mathcal{O}\left(\alpha_{\mathrm{s}}^{N+1}(Q^2)\right) . \tag{6.16}$$

The power-behaving component contains several sources. The first is the truncation error in $\mathcal{R}^{\mathrm{merge}}$ which amounts to

$$\Delta_{\mathcal{R}}^{\mathrm{merge}} = \mathcal{O}\left(\alpha_{\mathrm{s}}(Q^2)\,\lambda^N\,\frac{\mu_I}{Q}\right) = \mathcal{R}^{\mathrm{NP}}\cdot\mathcal{O}\left(\frac{\alpha_{\mathrm{s}}}{\alpha_0}\lambda^N\right), \qquad \lambda \equiv \beta_0\frac{\alpha_{\mathrm{s}}}{2\pi}\ln\frac{Q}{\mu_I} < 1\,. \tag{6.17}$$

The λ parameter is smaller than unity provided that $\mu_I > \Lambda_{\mathrm{QCD}}$. It is important to stress that these estimates are uniform in N, since the factorially growing coefficients in $\mathcal{R}^{\mathrm{PT}}$ and $\mathcal{R}^{\mathrm{merge}}$ cancel. Constructing the difference of the corresponding renormalon contributions, for the remainder of the series for $\mathcal{R}^{\mathrm{PT}}(Q,N)-\mathcal{R}^{\mathrm{merge}}(Q,\mu_I;N)$ we have

$$\alpha_{\mathrm{s}}\left(\beta_0\frac{\alpha_{\mathrm{s}}}{2\pi}\right)^N\int_0^{\ln\frac{Q}{\mu_I}} dt\, t^N\, e^{-t} \simeq \alpha_{\mathrm{s}}(Q^2)\lambda^N\,\frac{\ln(Q/\mu_I)}{(N+1)}\,\frac{\mu_I}{Q} \ll \Delta_{\mathcal{R}}^{\mathrm{merge}}, \qquad \text{for } N \gg \ln\frac{Q}{\mu_I}\,.$$

The renormalon leftover being smaller than (6.17) prevents the magnitudes of the errors in (6.16) and (6.17) from exploding at large N.

Another error originates from the cutoff on the momentum integral in (6.4) at the finite value μ_I:

$$\Delta_{\mathcal{R}}^{\mathrm{NP}} = \mathcal{O}\left(\alpha_{\mathrm{s}}^{\mathrm{NP}}(\mu_I^2)\frac{\mu_I}{Q}\right) = \mathcal{R}^{\mathrm{NP}}\cdot\mathcal{O}\left(\frac{\alpha_{\mathrm{s}}^{\mathrm{NP}}(\mu_I^2)}{\alpha_0}\right). \tag{6.18}$$

Within the logic of the present approach, $\alpha_{\mathrm{s}}^{\mathrm{NP}}$ is a rapidly falling function, so that by choosing a sufficiently large μ_I one can make this contribution arbitrarily small.

The largest contribution to the error in the power-behaved piece will of course come from higher corrections to the Milan factor which will have an effect of the form

$$\mathcal{M} \Longrightarrow \left(1+\mathcal{O}\left(\frac{\alpha_{\mathrm{s}}}{\pi}\right)\right)\mathcal{M}\,,$$

where α_{s} enters at some small scale. To quantify such effects one has to go beyond a two-loop calculation. Whether they are reasonably small depends on the effective interaction strength at small scales. The present-day phenomenological estimate of $\alpha_{\mathrm{s}}/\pi \simeq 0.2$ may lead one to be hopeful.

To conclude our discussion of the errors, we note that the difference $(\mathcal{R}^{\mathrm{PT}}-\mathcal{R}^{\mathrm{merge}})$, the "regularised" perturbative contribution to the observable, essentially corresponds to the introduction of an infrared cutoff μ_I in the frequencies in Feynman diagrams. The regularised perturbative contribution, as expected, depends on the infrared cutoff μ_I.

The μ_I-dependences in $\mathcal{R}^{\rm NP}(\mu_I)$ and $\mathcal{R}^{\rm merge}(Q,\mu_I;N)$ compensate each other. With increasing N, the residual μ_I-dependence disappears. One can verify this by evaluating the derivative of (6.15) with respect to μ_I:

$$\frac{\partial}{\partial \ln \mu_I}\mathcal{R}(Q) \;=\; \mathcal{O}\left(\Delta_{\mathcal{R}}^{\rm merge}+\Delta_{\mathcal{R}}^{\rm NP}\right).$$

A final remark concerns the scheme dependence. Explicit expressions for the series for both $\mathcal{R}^{\rm PT}(Q,N)$ and $\mathcal{R}^{\rm merge}(Q,\mu_I;N)$ depend on the scheme, that is on the choice of the perturbative expansion parameter $\alpha_{\rm s}$. Throughout our analysis we have been using the so-called physical (CMW) scheme [15, 16] in which the intensity of soft gluon radiation equals, by definition, $\alpha_{\rm s}$. Shifting, for example, to the popular $\overline{\rm MS}$ scheme amounts to substituting

$$\alpha_{\rm s} \Longrightarrow \alpha_{\overline{\rm MS}} + K\frac{\alpha_{\overline{\rm MS}}^2}{2\pi} + \ldots \tag{6.19}$$

in the perturbative series, with K as defined in (3.21). As for the non-perturbative parameter α_0 (and α_0', see below) it is *defined* in (6.6) in terms of the CMW coupling. Changing the scheme would change the *expression* in the left-hand side of (6.6), but would not affect the *value* of α_0.

Notice that the normalisation of the second, power behaving, term in (6.15) namely,

$$\mathcal{R}^{\rm NP}(Q,\mu_I) - \mathcal{R}^{\rm merge}(Q,\mu_I;N)\,,$$

differs by the factor $2\mathcal{M}/\pi$ from that of [2] which is being currently used in the experimental analyses. We return to this point in the conclusions.

6.1. $N=2$ merging

For the practically important case $N=2$, when the perturbative answer is known at the second order in the coupling constant $\alpha_{\rm s}\equiv\alpha_{\rm s}(Q^2)$, the relevant moments determining the power contributions to T, D, C and B radiators read

$$A_1 \simeq \frac{2C_F}{\pi^2}\mu_I\left\{\alpha_0(\mu_I)-\alpha_{\rm s}-\beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I}+1\right)\right\}, \tag{6.20}$$

and

$$\tfrac{1}{2}A_1' + A_1 \simeq \frac{2C_F}{\pi^2}\mu_I\left\{\alpha_0'(\mu_I)+\alpha_{\rm s}+\beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I}+2\right)\right\}. \tag{6.21}$$

Here the non-perturbative parameter α_0 has been defined in (6.6), and α_0' is given by a similar integral but with an extra logarithmic factor:

$$\alpha_0(\mu_I)\equiv\int_0^{\mu_I}\frac{dk}{\mu_I}\,\alpha_{\rm s}(k^2)\,,\qquad \alpha_0'(\mu_I)\equiv\int_0^{\mu_I}\frac{dk}{\mu_I}\,\alpha_{\rm s}(k^2)\,\ln\frac{k}{\mu_I}\,. \tag{6.22}$$

Let us note that from (6.22) we expect the new non-perturbative parameter α_0' to be *negative*. Using the dispersive representation (3.22), it is straightforward to derive the following relation for $\frac{1}{2}A_1' + A_1$

$$\tfrac{1}{2}A_1' + A_1 = \frac{2C_F}{\pi^2}\int_0^\infty dk\,\alpha^{\rm NP}(k^2)\,\ln\frac{k}{\mu}\,. \tag{6.23}$$

In analogy with the treatment of the A_1 moment, (6.4), we truncate the integral on the right-hand side at the matching scale, μ_I:

$$\frac{2C_F}{\pi^2}\int_0^\infty dk\,\alpha^{\rm NP}(k^2)\,\ln\frac{k}{\mu} \approx \frac{2C_F}{\pi^2}\int_0^{\mu_I} dk\,\alpha^{\rm NP}(k^2)\,\ln\frac{k}{\mu}\,.$$

Linear observables. Thus, the merging contribution to $V = 1–T, D, C$ distributions is, to second order,

$$\mathcal{R}^{\rm merge}_{(V)}(Q,\mu_I;2) = \rho^{(V)}\cdot\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\left(\alpha_{\rm s} + \beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I}+1\right)\right), \tag{6.24}$$

and the final answer takes the form

$$\mathcal{R} = \mathcal{R}^{\rm PT}(Q;2) + \rho^{(V)}\cdot\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\left[\alpha_0 - \alpha_{\rm s} - \beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I}+1\right)\right]. \tag{6.25}$$

Log-Linear observable. Substituting (6.20) into the non-perturbative broadening radiator (4.20), we arrive at

$$\mathcal{R}^{\rm NP}_{(B)} = 2\nu\,\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\left[\alpha_0\left(\ln\frac{Q}{\mu_I}-\xi\right)-\alpha_0'\right]; \tag{6.26}$$

$$\mathcal{R}^{\rm merge}_{(B)} = 2\nu\,\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\left[\alpha_{\rm s}\left(\ln\frac{Q}{\mu_I}-\xi+1\right)+\beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln^2\frac{Q}{\mu_I}+(2-\xi)\left(\ln\frac{Q}{\mu_I}+1\right)\right)\right]. \tag{6.27}$$

The final answer takes the form

$$\begin{aligned}\mathcal{R}_{(B)} = \mathcal{R}^{\rm PT} + 2\nu\,\frac{4C_F}{\pi^2}\,\mathcal{M}\,\frac{\mu_I}{Q}\Bigg[&\left(\ln\frac{Q}{\mu_I}-\xi\right)\left(\alpha_0-\alpha_{\rm s}-\beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I}+1\right)\right) - \\ &-\left(\alpha_0'+\alpha_{\rm s}+\beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I}+2\right)\right)\Bigg].\end{aligned} \tag{6.28}$$

7. Summary and discussion

In this paper we have considered the group of event shape observables that exhibit $1/Q$ non-perturbative corrections. We have shown that the two-loop effects result in a rescaling of the "naive" perturbative estimate of the magnitude of the power terms by a *universal* Milan factor. This is true for the non-perturbative effects in the thrust, in the invariant jet mass and in the C-parameter distributions, and for the main $\ln Q$–enhanced power term in jet broadening. The same Milan factor also applies to the energy-energy correlation measure away from the back-to-back region.

The universality of the Milan factor for the linear jet shape observables in DIS has recently been demonstrated by M. Dasgupta and B.R. Webber in [25].

We have shown, following [19] that the $1/Q$ power effects in jet shape distributions result in a *shift* of the perturbative spectra,

$$\frac{d\sigma}{dV}(V) = \left[\frac{d\sigma}{dV}\left(V - c_V\mathcal{P}\right)\right]_{\rm PT} \qquad V = 1-T,\, C,\, \frac{M_T^2}{Q^2},\, \frac{M_H^2}{Q^2}\,; \tag{7.1}$$

$$\frac{d\sigma}{dV}(V) = \left[\frac{d\sigma}{dV}\left(V - c_V\mathcal{P}\ln\frac{Q}{Q_B}\right)\right]_{\rm PT} \qquad V = B_T,\, B_W\,. \tag{7.2}$$

The all-order resummed expressions for the perturbative spectra in (7.1) can be found, for the thrust and jet-masses, in [11], and for the C-parameter in [26]. For the perturbative broadening spectrum (7.2) one should use the recent theoretical prediction [14], which improves the treatment of subleading effects compared to the original derivation [12].

The answers are valid in the region $1 \gg V \gg \mathcal{P} \sim \Lambda_{\rm QCD}/Q$. The relative coefficients c_V are given in the following table.

$$\begin{array}{|c|c||c||c|c||c|c|} \hline V = & 1-T & C & M_T^2 & M_H^2 & B_T & B_W \\ \hline c_V = & 2 & 3\pi & 2 & 1 & 1 & \frac{1}{2} \\ \hline \end{array} \tag{7.3}$$

Note that the power shifts in the single-jet characteristics (M_H^2, B_W) amount to half of those in the total distributions (M_T^2, B_T).

The non-perturbative parameter $\mathcal{P}$ was defined in (4.11). The B-distribution contains an additional non-perturbative parameter, the scale of the logarithm Q_B, related to $\mathcal{P}'$, (4.24), the first log-moment of the "coupling modification" $\delta\alpha_{\rm eff}$. However, as was discussed in section 6, the parameters $\mathcal{P}$ and $\mathcal{P}'$ (or, A_1, A_1') suffer from the same infrared renormalon ambiguity as the perturbative series. To remove this ambiguity one has to introduce the "infrared matching" momentum scale μ_I. At the two-loop level, the expressions for the shifts in terms of the non-perturbative μ_I-dependent phenomenological parameters α_0 (6.6) and α_0' (6.22) are:

$$\mathcal{P} \simeq \frac{4C_F}{\pi^2}\mathcal{M}\frac{\mu_I}{Q}\left\{\alpha_0(\mu_I) - \alpha_{\rm s} - \beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I} + \frac{K}{\beta_0} + 1\right)\right\}\,, \tag{7.4}$$

$$\mathcal{P}' \simeq \frac{4C_F}{\pi^2}\mathcal{M}\frac{\mu_I}{Q}\left\{\alpha_0'(\mu_I) + \alpha_{\rm s} + \beta_0\frac{\alpha_{\rm s}^2}{2\pi}\left(\ln\frac{Q}{\mu_I} + \frac{K}{\beta_0} + 2\right)\right\}\,; \quad \alpha_{\rm s} \equiv \alpha_{\overline{\rm MS}}(Q^2)\,. \tag{7.5}$$

Here we have given the $\overline{\rm MS}$ expressions, with K defined in (3.21). The parameter $\mathcal{P}'$ enters into the shift in the broadening distribution (7.2) as

$$\mathcal{P}\ln\frac{Q}{Q_B} = \mathcal{P}\left(\ln\frac{Q}{\mu_I} - \xi\right) - \mathcal{P}'\,, \tag{7.6}$$

with the numerical parameter ξ defined in (4.21). Recall that we expect the parameter α_0' to be negative if the QCD coupling is to remain positive in the infrared region, see (6.22). The $\ln Q$–enhancement of the $1/Q$ contribution to jet broadening remains to be verified experimentally.

By now there is a good experimental evidence for $1/Q$ power terms in jet shape variables both in e^+e^- [27] and DIS [28]. The magnitudes are consistent, within a 20% margin, for different observables. An estimate of the basic A_1 parameter can be obtained, for example, from the second order fit to the Q-dependence of the mean thrust,

$$\langle 1-T\rangle = a_1\alpha_s + a_2\alpha_s^2 + 2\mathcal{P}\,, \tag{7.7}$$

with a_1, a_2 the known perturbative coefficients. If one boldly used the original relation (4.11), $\mathcal{P} = \mathrm{const}/Q$, the fit to the data would produce $A_1 \simeq 0.15\,\mathrm{GeV}$. However, such an estimate is meaningless: the very representation (7.7) would be renormalon-infested. Therefore one should use instead in (7.7) the representation (7.4) for $\mathcal{P}$ which is constructed as the difference of the pure power contribution, $\mathcal{R}^{\mathrm{NP}}$, and the perturbative subtraction, $\mathcal{R}^{\mathrm{merge}}$. Thus, the price one pays for avoiding the renormalon ambiguity is the introduction of the finite infrared "matching scale" $\mu_I/\Lambda_{\mathrm{QCD}}$ which enters into the shifts (7.4), (7.5) both explicitly and via α_0 and α_0'. As we have argued above in section 6, the μ_I-dependence gets weaker when higher orders of the perturbative expansion are included. The accuracy of the merging of the perturbative and non-perturbative contributions is of order

$$\frac{\mu_I}{Q}\left(\beta_0\frac{\alpha_s(Q^2)}{2\pi}\ln\frac{Q}{\mu_I}\right)^{N+1},$$

with N the order of the highest perturbative term included ($N\!=\!2$ in (7.7)). A reasonable value to take for μ_I is $\mu_I = 2\,\mathrm{GeV}$.

Having considered the problem of the infrared renormalon divergence of the perturbative series, we have chosen to quantify the non-perturbative effects in terms of the momentum integrals of the running coupling $\alpha_s(k^2)$ rather than in terms of the moments of the non-perturbative effective coupling modification, $\delta\alpha_{\mathrm{eff}}(m^2)$, as was suggested in [2]. Our non-perturbative parameter α_0 differs by a factor $2/\pi$ from the analogous parameter $\bar{\alpha}_0$ of [2]. This factor originates from the "translation" relation

$$\frac{2}{\pi}\int_0^\infty dk\,\alpha_s^{\mathrm{NP}}(k^2) = \int_0^\infty dm\,\delta\alpha_{\mathrm{eff}}(m^2)\,.$$

The power contributions $(\mathcal{R}^{\mathrm{NP}} - \mathcal{R}^{\mathrm{merge}})$ of [2, 19] were written in terms of $\bar{\alpha}_0$ and did not include the two-loop effects (the Milan factor). Thus, to switch to the expressions of this paper (based on α_0, α_0'), they should be multiplied by $2/\pi\cdot\mathcal{M}$. These two factors practically compensate each other: $2\mathcal{M}/\pi \simeq 1.14$ ($n_f = 3$). Strictly speaking, the remaining small renormalisation could affect the phenomenological fits to the data, since it multiplies both the non-perturbative term $\mathcal{R}^{\mathrm{NP}}$ *and* the perturbative subtraction contribution ($\mathcal{R}^{\mathrm{merge}}$), the latter possessing a residual $\log Q$–dependence.

We could have chosen to express the answer in terms of the moments of α_{eff} as suggested in [2]. This would have required, for the sake of consistency, the reformulation also of the standard PT series and the PT subtraction in terms of α_{eff}. Carried out to all orders, this would have given the same answer. However, if we restrict ourselves to $N = 2$ we observe the following puzzle: neither $\mathcal{R}^{\text{PT}}$ nor $\mathcal{R}^{\text{merge}}$ is sensitive to the difference between α_{s} and α_{eff} to this order. As a result the coefficients of $\mathcal{R}^{\text{merge}}(\alpha_{\text{eff}})$ and $\mathcal{R}^{\text{merge}}(\alpha_{\text{s}})$ differ by the factor $2/\pi$, apparently for no good reason. This mismatch can be absorbed, but only partially, if one chooses an appropriate relation between the cutoff scales μ_I in the two representations. We believe that the origin of the puzzle lies in the fact that the relation between α_{s} and α_{eff} intrinsically contains the non-perturbative region, and so only holds when one considers all orders of the perturbation series. We intend to consider this problem in more detail in a future publication.

Concluding the discussion of the Milan factor we should stress that only after having gone through the ordeal of the two-loop analysis, can one unambiguously predict the relative magnitudes of the power-behaving confinement contributions to different observables. This is because the corrections to the magnitude of the power term which are quadratic in small-scale α_{s}, get promoted to a finite renormalisation factor. This can be understood with the help of the following qualitative argument. At the perturbative level, the α_{s}^2 correction can be recast in terms of a rescaling of Λ_{QCD} in the argument of α_{s} in the one-loop answer. At the level of the power terms, however, such a rescaling of the argument of the coupling immediately triggers a corresponding rescaling of the dimensionful *numerator* of the $1/Q$ contribution. In spite of the entirely illustrative nature of the argument, it gives a hint as to why we do not expect the higher loop corrections to be as crucially important as the two-loop one. Indeed corrections beyond the second loop do not affect the characteristic momentum scale of the problem and thus the normalisation of the power contributions acquires only a factor of the form $1 + \mathcal{O}(\alpha_{\text{s}})$, which is formally subleading. Whether the relative size of such higher order corrections can be kept under control depends on the actual strength of the effective interaction at small scales and remains to be quantified.

Given the importance of the two-loop effects, the "naive" first order approach becomes intrinsically ambiguous. Within this treatment one introduces into the Feynman diagrams a coupling that runs with the gluon virtuality (fake gluon mass) in order to define a "projection" onto the confinement interaction region. Such a procedure is non-unique with respect to the precise determination of the argument of the coupling, and the inclusion of the finite gluon mass effects into the kinematics and into the definition of the observables. For example, in the definition of T for the ($q\bar{q}+$ massive gluon) system suggested by Beneke and Braun in [18], the finite mass effects were included in the normalisation of thrust ($p_q + p_{\bar{q}} + |\vec{k}_g| < Q$). Being as good a naive approximation as any other, this produces an answer which differs by a numerical factor from the "standard" one. Our claim is that if the two-loop effects were evaluated, the final result based on the Beneke-Braun prescription would coincide with the one obtained in [8] and generalised in the present paper.

Beyond the naive treatment, one can no longer embody all gluon decays into the running of the coupling. Those decay configurations that affect the value of the observable under consideration cannot be treated inclusively [7]. Though formally at the level α_s^2 they are still essential, as we know, since they produce the same leading $1/Q$ effects. This is the Nason-Seymour problem of non-inclusiveness of jet observables and it is a part of the whole programme of giving unambiguous predictions for the magnitudes of power-behaving contributions.

Our solution can be formulated as follows. First we give a definite prescription for the naive contribution which treats gluon decays inclusively and incorporates the running coupling and finite gluon mass effects. To this end we have chosen to substitute $\sqrt{k_t^2+m^2}$ for k_t both in the kinematics and in the definition of the observable, so as to treat a "massive gluon" contribution. Given this prescription, we have demonstrated that the two-loop corrections to the naive treatment amount to a universal, that is *observable-independent*, normalisation effect (Milan factor).

The universality of the Milan factor has three ingredients. Firstly, it relies on universality of soft radiation, the latter being responsible for confinement effects. Secondly, it is based on the concept of universality of the QCD interaction strength, all the way down to small momentum scales, which has been processed through the machinery of the dispersive approach. Finally, it includes a certain geometric universality of the observables under consideration. To clarify the last point, we note that the contribution of a massless parton i to a given observable V (linear in the limit of small parton momenta) can be factorised as

$$v_i = k_{ti} \cdot f^{(V)}(\eta_i)\,,$$

with k_{ti} the modulus of transverse momentum and η_i the rapidity of the parton, and f a function that depends on the observable. For example, for the thrust and C parameter we have

$$f^{(T)}(\eta) = e^{-\eta}\Theta(\eta) + e^{\eta}\Theta(-\eta)\,, \qquad f^{(C)}(\eta) = \frac{3}{\cosh\eta}\,.$$

The rapidity integral of this function enters linearly into the magnitude of the power contribution and determines the observable-dependent coefficient of the latter,

$$\rho^{(V)} = \nu \cdot \int d\eta\, f^{(V)}(\eta)\,.$$

The crucial observation is that this very same integral appears both in the terms driven by a parent gluon (naive $\mathcal{R}_0$ and the inclusive correction $\mathcal{R}^{in}$) as well as in the separate contributions from offspring partons the gluon decays into (the non-inclusive correction $\mathcal{R}^{ni}$). The two decay partons give

$$v_1 + v_2 = k_{t1} f^{(V)}(\eta_1) + k_{t2} f^{(V)}(\eta_2)\,,$$

which contribution has to be weighted with the parent gluon decay probability. First we observe that the decay matrix element depends only on the relative rapidity $\Delta\eta = \eta_1 - \eta_2$ (Lorentz invariance). Then, that $\Delta\eta$ stays finite in the essential region (IR-safety of V). Therefore, keeping $\Delta\eta$ and k_{ti} fixed, we can perform an overall rapidity integration extending it to infinity:

$$v_1 + v_2 \Longrightarrow (k_{t1} + k_{t2}) \cdot \rho^{(V)} .$$

As a result, the dependence on the observable factors out thus ensuring the universality of the non-inclusive correction. What is left is the standard integral of the sum of two transverse momenta weighted with the two-parton decay matrix element.

For some specific observables such as the height of the transverse momentum distribution of Drell-Yan pairs at $q_t \to 0$, EEC with back-to-back kinematics, etc. we expect an interplay between the perturbative resummation and the genuine confinement effects to lead to calculable *fractional* powers of Q in the non-perturbative correction terms. Whether this expectation is correct is an intriguing question which should be addressed both theoretically and experimentally. The magnitude of these fractional power contributions is determined by the non-perturbative parameters A_1 and $A_{2,1}$ (see (4.1)). The latter parameter enters also in the $1/Q^2$ power corrections to DIS (e^+e^-) structure (fragmentation) functions, DIS sum rules, the Drell-Yan K-factor, as well as in the mean value of the 3-jet resolution parameter y_3 [6, 29]. From the analysis of F_2, Dasgupta and Webber [29] derived a phenomenological estimate $A_{2,1} \simeq -0.2$ GeV^2. This implies that confinement effects should *dampen* the height of the Drell-Yan distribution at $q_t = 0$.

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Erratum

The results of [1] are in part based on those of [2] which following an independent calculation [3] has been found to contain a mistake related to a factor of two in the phase space for one of the components of the calculation. The mistake has already been briefly discussed and corrected in [4], and the full details of the correction to the original calculation can be found in the erratum to [2]. The modifications that should be made to [1] in the light of these corrected results are that eqs. (4.8) and (4.10) should be changed to read

$$\delta\mathcal{R}_{ni}^{(T,D,C)} = \delta\mathcal{R}_0^{(C,D,C)}\, r_{ni}\,, \qquad r_{ni} = 2\beta_0^{-1}\left(-1.227C_A + 0.365C_A - 0.052n_f\right), \tag{4.8}$$

and

$$\begin{aligned}\mathcal{M} &= 1 + r_{in} + r_{ni} = 1 + \beta_0^{-1}\left(1.575C_A - 0.104n_f\right)\\ &= 1.490(1.430) \qquad \text{for } n_f = 3(0)\,.\end{aligned} \tag{4.10}$$

We also point out that the analysis of the broadenings in [1] is superseded by that which appears in [5].

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Added references

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